

ON CALCULUS OF FUNCTORS IN MODEL CATEGORIES

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ABSTRACT. We present an analysis of some constructions and arguments from the universe of T. G. Goodwillie's Calculus, in a general model theoretic setting.

In this paper we undertake a small analysis of T. G. Goodwillie's Calculus [4],[5] in model categories. To the best of our knowledge, it was N. Kuhn [9] who first described a general model theoretic setting for the development of Calculus and outlined Goodwillie's main results from [5] in this setting. Our work is influenced by his, but it has a somewhat different purpose. Our analysis is mainly oriented towards making more conceptual some of the constructions and arguments from the universe of Calculus, while retaining a general model theoretic setting. This setting differs slightly from Kuhn's. We hope that the analysis also reveals some of the complications that might arise when one wants to work in such an environment.

The first topic we address in our analysis is the construction of the Taylor tower. Let \mathbf{Cat} be the category of all small categories. Starting with a small subcategory \mathbf{J} of \mathbf{Cat} which does not contain the empty category we construct a sequence $\{\mathbf{J}(n+1)\}_{n \geq 0}$ of small subcategories of \mathbf{Cat} . $\mathbf{J}(n+1)$ is obtained from $\mathbf{J}(n)$ using a Grothendieck construction. When \mathbf{J} consists of the terminal object of \mathbf{Cat} only, $\mathbf{J}(n+1)$ is the set of subsets of the set $\{1, 2, \dots, n+1\}$ (viewed as a preorder with inclusions as arrows) with the empty set removed. To each functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between simplicial model categories and each small subcategory \mathbf{J} of \mathbf{Cat} which does not contain the empty category, we associate a tower of functors $\{P_{n,\mathbf{J}}F : \mathcal{C} \rightarrow \mathcal{D}\}_{n \geq 0}$ which coincides with Goodwillie's Taylor tower $\{P_n F\}_{n \geq 0}$ of F [5] when \mathbf{J} consists of the terminal object of \mathbf{Cat} only.

The second topic is the proof of Goodwillie's n -excisive approximation theorem [5, Theorem 1.8]. To each functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between simplicial model categories we associate a tower of functors $\{\tilde{P}_n F : \mathcal{C} \rightarrow \mathcal{D}\}_{n \geq 0}$. $\tilde{P}_n F$ is a minor variation of $P_n F$. There is a natural map of towers $\{\tilde{P}_n F\} \rightarrow \{P_n F\}$, and for suitable \mathcal{C} and F this map is a weak equivalence when evaluated at cofibrant objects. We believe that the use of the tower $\{\tilde{P}_n F\}$ sheds some light into Rezk's proof [11] of Goodwillie's theorem.

The third topic is concerned with the notions of (strongly) homotopy (co-)Cartesian cubes. In the first place, there is a body of technical results, due to Goodwillie and others, related to homotopy Cartesian cubes. We introduce generalized homotopy Cartesian cubes, and we prove the analogue, in our context, of some of these results. In the second place, we give a model theoretic interpretation of the notions of homotopy co-Cartesian and strongly homotopy co-Cartesian cube.

Notation. We denote by \mathbf{Cat} the category of all small categories and by CAT that of all large categories. For $\mathcal{C}, \mathcal{D} \in CAT$, we denote by $Fun(\mathcal{C}, \mathcal{D})$ the category of functors $\mathcal{C} \rightarrow \mathcal{D}$ and natural transformations between them. When \mathcal{C} is small, we shall use the notation $\mathcal{D}^{\mathcal{C}}$ instead. The terminal (initial, zero, respectively) object of a category, when it exists, is denoted by $*$ (\emptyset , 0 , respectively). We let $[1]$ be the category freely generated by the graph $\{0 \rightarrow 1\}$, pb the category freely generated by the graph $0 \rightarrow 1 \leftarrow 2$, po the category freely generated by the graph $0 \leftarrow 1 \rightarrow 2$ and ω the free category generated by the graph $\{0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow \dots\}$. If S is a set, $|S|$ is its cardinal. We denote by \mathbf{S} the category of simplicial sets and by N the nerve functor from \mathbf{Cat} to \mathbf{S} .

1. SOME GROTHENDIECK CONSTRUCTIONS

1.1. Contravariant Grothendieck constructions. Let \mathbb{B} be a small category and $\Psi : \mathbb{B}^{op} \rightarrow \mathbf{Cat}$ a functor. Recall that the (contravariant) **Grothendieck construction** $\int_{\mathbb{B}}(\Psi)$ (or simply $\int_{\mathbb{B}}\Psi$) of Ψ is the category with objects (b, x) where $b \in \mathbb{B}$ and $x \in \Psi(b)$, and arrows $(b, x) \rightarrow (b', y)$ are pairs (u, f) with $u : b \rightarrow b'$ in \mathbb{B} and $f : x \rightarrow \Psi(u)(y)$ in $\Psi(b)$.

The projection $p : \int_{\mathbb{B}}(\Psi) \rightarrow \mathbb{B}$ is a split fibration. The fiber category of p over $b \in \mathbb{B}$ is isomorphic to $\Psi(b)$. For each $b \in \mathbb{B}$, we denote by τ_b the natural functor from the fiber category of p over b to $\int_{\mathbb{B}} \Psi$.

One has the formula

$$\int_{\mathbb{B}} \Psi \cong \int_{\mathbb{B}} \Psi(b) \times (\mathbb{B} \downarrow b) \quad (1)$$

Example 1.1. Let J be a small category. We denote by J_+ the category obtained by adding an initial object \emptyset to J . One has $J_+ = \int_{[1]} \Psi$, where $\Psi(1) = J$ and $\Psi(0) = *$. The natural functor $\tau_1 : J \rightarrow J_+$ is an inclusion. Writing the coend in formula (1) as a colimit we have a pushout diagram

$$\begin{array}{ccc} J & \xrightarrow{\quad} & * \\ \downarrow & & \downarrow \\ J \times [1] & \xrightarrow{\quad} & \int_{[1]} \Psi \end{array}$$

where the right vertical arrow sends $j \in J$ to $(j, 0)$.

Example 1.2. Let $F : I \rightarrow J$ be a functor between small categories. Let $\Psi(F) : pb^{op} \rightarrow \mathbf{Cat}$ be the functor

$$\begin{array}{ccc} & I = \Psi(1) & \\ F \swarrow & & \searrow \\ J = \Psi(0) & & * = \Psi(2) \end{array}$$

One may then form $\int_{pb} \Psi(F)$. We shall use this construction when F is the identity functor of a small category J . In this case we denote $\int_{pb} \Psi(Id_J)$ by $\int_{pb} J$. We have natural functors

$$J \xrightleftharpoons[\tau_1]{\tau_0} \int_{pb} J$$

Writing the coend in formula (1) as a colimit we have a pushout diagram

$$\begin{array}{ccc} J & \xrightarrow{\quad} & * \\ \downarrow & & \downarrow \\ J \times pb & \xrightarrow{\quad} & \int_{pb} J \end{array}$$

where the left vertical arrow sends $j \in J$ to $(j, 2)$. This pushout can be calculated as the pushout

$$\begin{array}{ccc} J & \xrightarrow{\quad} & J_+ \\ \downarrow & & \downarrow \\ J \times [1] & \xrightarrow{\quad} & \int_{pb} J \end{array} \quad (2)$$

where the left vertical arrow sends $j \in J$ to $(j, 1)$. One has

$$\left(\int_{pb} J \right)_+ \cong J_+ \times [1] \quad (3)$$

Construction 1.3. Given a small category J , we construct by induction a sequence $\{J(n+1)\}_{n \geq 0}$ of small categories, as follows. $J(1) = J$ and

$$J(n+1) = \int_{pb} J(n)$$

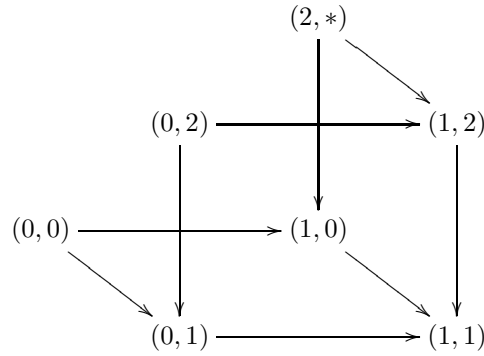
The natural functors (Example 1.2) $\tau_0, \tau_1 : J(n) \rightarrow \int_{pb} J(n)$ shall be denoted by τ_0^n and τ_1^n , so that we have a sequence

$$J = J(1) \xrightleftharpoons[\tau_1^1]{\tau_0^1} J(2) \xrightleftharpoons[\tau_1^2]{\tau_0^2} J(3) \xrightleftharpoons[\tau_1^3]{\tau_0^3} \dots$$

From Example 1.2 we have

$$J(n+1)_+ \cong J(n)_+ \times [1]$$

Example 1.4. $*(3)$ is



and this has already been observed in [2, Example 38.2] (and perhaps other places). In general, let S be a finite set. We denote by $\mathcal{P}(S)$ the set of all subsets of S . $\mathcal{P}(S)$ is a preorder under inclusion. We let $\mathcal{P}_0(S) = \mathcal{P}(S) - \{\emptyset\}$ and $\mathcal{P}_1(S) = \mathcal{P}(S) - \{S\}$. For $n \geq 0$ we let $\underline{n} = \{1, 2, \dots, n\}$ and $\underline{0} = \emptyset$. One has

$$*(n+1) \cong \mathcal{P}_0(\underline{n+1}) \quad (n \geq 0)$$

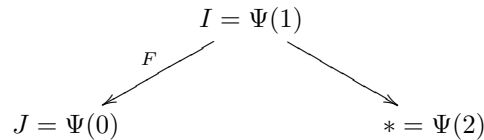
The maps $\tau_0^n, \tau_1^n : \mathcal{P}_0(\underline{n}) \rightarrow \mathcal{P}_0(\underline{n+1})$ are $\tau_0^n(S) = S$ and $\tau_1^n(S) = S \cup \{n+1\}$.

1.2. Covariant Grothendieck constructions. Let \mathbb{B} be a small category and $\Psi : \mathbb{B} \rightarrow \mathbf{Cat}$ a functor. Recall that the covariant **Grothendieck construction** $\int_{\mathbb{B}} \Psi$ of Ψ is the category with objects (b, x) where $b \in \mathbb{B}$ and $x \in \Psi(b)$, and arrows $(b, x) \rightarrow (b', y)$ are pairs (u, f) with $u : b \rightarrow b'$ in \mathbb{B} and $f : \Psi(u)(x) \rightarrow y$ in $\Psi(b')$. The projection $p : \int_{\mathbb{B}} \Psi \rightarrow \mathbb{B}$ is a split opfibration. The fiber category of p over $b \in \mathbb{B}$ is isomorphic to $\Psi(b)$.

As a rather trivial example, let S be a nonempty set, viewed as a discrete category. The category S_+ (Example 1.1) is (also) a covariant Grothendieck construction. Indeed, let $2_{|S|}$ be the category with two objects 0 and 1 and $|S|$ arrows from 0 to 1. Let $\Psi : 2_{|S|} \rightarrow \mathbf{Cat}$ be defined as $\Psi(0) = *$ and $\Psi(1) = S$. Then $S_+ = \int_{2_{|S|}} \Psi$.

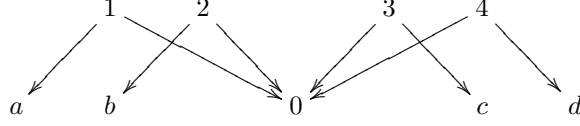
More interesting is the dual of Example 1.2.

Example 1.5. Let $F : I \rightarrow J$ be a functor between small categories. Let $\Psi(F) : po \rightarrow \mathbf{Cat}$ be the functor



One may then form the covariant Grothendieck construction $\int_{po} \Psi(F)$. We shall use this construction when F is the identity functor of a small category J . In this case we denote $\int_{po} \Psi(Id_J)$ by $\int_{po} J$. When J is the discrete category on

a set S , $\int_{po} S$ maybe depicted as a spider with $|S|$ legs. We illustrate the case $S = \underline{4}$ (Example 1.4)



We call the object $(2, *)$ of $\int_{po} S$ the body of the spider (0 in the above picture).

Let J be a small category, \mathcal{C} a cocomplete category and $\mathcal{Z} : (J_+)^{op} \times J_+ \rightarrow \mathcal{C}$ a functor. For later purposes we need to understand $\int_{j \in J_+} \mathcal{Z}(j, j)$ better, especially when J is a discrete category. To begin with, we recall the calculation of $\int_{j \in J_+} \mathcal{Z}(j, j)$ as a colimit. The (lower) **twisted arrow category** of J , which we denote by J_τ , has arrows $f : j \rightarrow k$ of J as objects and a map $f \rightarrow f'$ is a commutative diagram

$$\begin{array}{ccc} j & \xrightarrow{\quad} & j' \\ f \downarrow & & \downarrow f' \\ k & \xleftarrow{\quad} & k' \end{array}$$

in J . There is a functor $K : (J_+)_\tau \rightarrow (J_+)^{op} \times J_+$, $K((j \rightarrow k)) = (k, j)$, and

$$\int_{j \in J_+} \mathcal{Z}(j, j) \cong \operatorname{colim}_{(J_+)_\tau} \mathcal{Z} K$$

Now, if J is a discrete category, there is a natural isomorphism

$$\xi_J : \int_{po} J \xrightarrow{\cong} (J_+)_\tau$$

given by $(0, j) \mapsto Id_j$, $(1, j) \mapsto (\emptyset \rightarrow j)$ and $(2, *) \mapsto Id_\emptyset$. Therefore we have a natural isomorphism

$$\operatorname{colim}_{\int_{po} J} \mathcal{Z} K \xi_J \xrightarrow{\cong} \operatorname{colim}_{(J_+)_\tau} \mathcal{Z} K$$

Lemma 1.6. (Inheritance results)

(a) If J is a small Reedy category with cofibrant constants then so are J_+ and $\int_{pb} J$.

(b) If \mathbf{J} is a small subcategory of \mathbf{Cat} (which does not contain the empty category), then $\int_{pb} \mathbf{J}$ is a small subcategory of \mathbf{Cat} (which does not contain the empty category).

Example 1.7. Continuing Example 1.4, $\mathcal{P}_0(\underline{n+1})$ is a Reedy category with cofibrant constants if we let the inverse subcategory $\overleftarrow{\mathcal{P}_0(\underline{n+1})}$ be $\mathcal{P}_0(\underline{n+1})$, the direct subcategory $\overrightarrow{\mathcal{P}_0(\underline{n+1})}$ be the discrete category on the set of object of $\mathcal{P}_0(\underline{n+1})$ and the degree of $S \in \mathcal{P}_0(\underline{n+1})$ be $n+1 - |S|$.

Lemma 1.6 has a dual formulation. In particular, for every finite set S , $\int_{po} S$ is a Reedy category with fibrant constants.

2. HOMOTOPY LIMITS AND COLIMITS I

In order to construct the Goodwillie tower of a functor between simplicial model categories, we recall in this section the minimum necessary from the theory of homotopy limits and colimits in simplicial model categories

2.1. Homotopy limits. Let \mathcal{C} be a category. We denote by $(\mathbf{Cat} // \mathcal{C})$ the category with objects pairs $(I, \mathcal{X} : I \rightarrow \mathcal{C})$, where $I \in \mathbf{Cat}$, and arrows $\langle F, \alpha \rangle : (I, \mathcal{X} : I \rightarrow \mathcal{C}) \rightarrow (J, \mathcal{Y} : J \rightarrow \mathcal{C})$ those pairs consisting of a functor $F : I \rightarrow J$ and a natural transformation $\alpha : \mathcal{Y}F \Rightarrow \mathcal{X}$. If \mathcal{D} is another category, a functor $f : \mathcal{C} \rightarrow \mathcal{D}$ induces a functor $(\mathbf{Cat} // f) : (\mathbf{Cat} // \mathcal{C}) \rightarrow (\mathbf{Cat} // \mathcal{D})$.

Let \mathcal{C} be a simplicial model category, J a small category and $\mathcal{X} : J \rightarrow \mathcal{C}$ a functor. $\text{holim}_J \mathcal{X}$ stands for the homotopy limit of \mathcal{X} , as defined in [6, 18.1.8]. holim is a functor $(\mathbf{Cat} // \mathcal{C})^{op} \rightarrow \mathcal{C}$. $\text{holim}_J : \mathcal{C}^J \rightarrow \mathcal{C}$ is a simplicial functor (for holim_J to be simplicial one does not need \mathcal{C}^J be a model category). A simplicial functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between simplicial model categories induces a natural map

$$F(\text{holim}_J \mathcal{X}) \rightarrow \text{holim}_J F\mathcal{X}$$

Let $\hat{F} : \mathcal{C} \rightarrow \mathcal{C}$ be a fibrant approximation on \mathcal{C} . cholim_J stands for the composite

$$\mathcal{C}^J \xrightarrow{\hat{F}} \mathcal{C}^J \xrightarrow{\text{holim}_J} \mathcal{C}$$

and is referred to as the corrected homotopy limit of J -diagrams in \mathcal{C} . cholim is a functor $(\mathbf{Cat} // \mathcal{C})^{op} \rightarrow \mathcal{C}$. We shall use both holim and cholim . If \mathcal{C} is cofibrantly generated, then \mathcal{C} has a simplicial fibrant approximation [6, 4.3.7], and, with this choice of fibrant approximation, cholim_J is a simplicial functor.

2.2. Homotopy colimits. Let \mathcal{C} be a category. We denote by $(\mathbf{Cat} \downarrow \mathcal{C})$ the category with objects pairs $(I, \mathcal{X} : I \rightarrow \mathcal{C})$, where $I \in \mathbf{Cat}$, and arrows $\langle F, \alpha \rangle : (I, \mathcal{X} : I \rightarrow \mathcal{C}) \rightarrow (J, \mathcal{Y} : J \rightarrow \mathcal{C})$ those pairs consisting of a functor $F : I \rightarrow J$ and a natural transformation $\alpha : \mathcal{X} \Rightarrow \mathcal{Y}F$. If \mathcal{D} is another category, a functor $f : \mathcal{C} \rightarrow \mathcal{D}$ induces a functor $(\mathbf{Cat} \downarrow f) : (\mathbf{Cat} \downarrow \mathcal{C}) \rightarrow (\mathbf{Cat} \downarrow \mathcal{D})$.

Let \mathcal{C} be a simplicial model category, J a small category and $\mathcal{X} : J \rightarrow \mathcal{C}$ a functor. $\text{hocolim}_J \mathcal{X}$ stands for the homotopy colimit of \mathcal{X} , as defined in [6, 18.1.2]. hocolim is a functor $(\mathbf{Cat} \downarrow \mathcal{C}) \rightarrow \mathcal{C}$. A simplicial functor $F : \mathcal{C} \rightarrow \mathcal{D}$ induces a natural map

$$\text{hocolim}_J F\mathcal{X} \rightarrow F(\text{hocolim}_J \mathcal{X})$$

Let $\tilde{C} : \mathcal{C} \rightarrow \mathcal{C}$ be a cofibrant approximation on \mathcal{C} . chocolim_J stands for the composite

$$\mathcal{C}^J \xrightarrow{\tilde{C}} \mathcal{C}^J \xrightarrow{\text{hocolim}_J} \mathcal{C}$$

and is referred to as the corrected homotopy colimit of J -diagrams in \mathcal{C} . chocolim is a functor $(\mathbf{Cat} \downarrow \mathcal{C}) \rightarrow \mathcal{C}$. We shall use both hocolim and chocolim .

3. THE CONSTRUCTION OF THE TAYLOR TOWER

In this section we examine the construction of the Taylor tower of a functor between simplicial model categories [5], [9]. After that we study the ingredients which appear in the construction.

3.1. The \star operation. We begin with some general considerations.

(1) Let \mathcal{V} be a monoidal category and \mathcal{C} a category equipped with an action $\otimes : \mathcal{V} \times \mathcal{C} \rightarrow \mathcal{C}$ of \mathcal{V} . Let A be a monoid in \mathcal{V} and Y a left A -module in \mathcal{C} . The category $(\mathcal{V} \downarrow A)$ becomes a monoidal category which acts on $(\mathcal{C} \downarrow Y)$. Let us denote this action by \otimes' . Suppose that \mathcal{C} is cocomplete. For each small category J we have an induced functor

$$\otimes'_J : (\mathcal{V} \downarrow A)^{J^{op}} \times (\mathcal{C} \downarrow Y)^J \longrightarrow (\mathcal{C} \downarrow Y)$$

(2) Let \mathcal{C} be a category with terminal object and J a small category. We define a functor $R_J : \mathcal{C} \rightarrow \mathcal{C}^{J+}$ as

$$R_J X(j) = \begin{cases} X, & \text{if } j = \emptyset \\ *, & \text{otherwise} \end{cases}$$

In particular, for any category \mathcal{C} and any $Y \in \mathcal{C}$ we have the functor $R_J : (\mathcal{C} \downarrow Y) \rightarrow (\mathcal{C} \downarrow Y)^{J+}$.

We shall apply the previous considerations in the following situation. \mathcal{C} is a simplicial model category, A is the terminal simplicial monoid and $Y \in \mathcal{C}$. For each small category J we have then a composite functor

$$\mathbf{S}_+^{J^{op}} \times (\mathcal{C} \downarrow Y) \xrightarrow{Id \times R_J} \mathbf{S}_+^{J^{op}} \times (\mathcal{C} \downarrow Y)^{J+} \xrightarrow{\otimes'_{J+}} (\mathcal{C} \downarrow Y)$$

Let $(X, X \rightarrow Y) \in (\mathcal{C} \downarrow Y)$. We define

$$J \star_Y X = \operatorname{hocolim}_{J_+} R_J(X, X \rightarrow Y)$$

We obtain a functor

$$- \star_Y - : \mathbf{Cat} \times (\mathcal{C} \downarrow Y) \rightarrow (\mathcal{C} \downarrow Y)$$

When $Y = *$ we denote $J \star_* X$ by $J \star X$, so that

$$J \star X = \operatorname{hocolim}_{J_+} R_J X$$

One has $\emptyset \star X = X$ and $* \star X = CX$, the cone on X . The \star operation can be restricted to any subcategory of \mathbf{Cat} which does not contain the empty category. In particular, for each $n \geq 0$ we have a functor $- \star - : \mathcal{P}_0(\underline{n+1}) \times \mathcal{C} \rightarrow \mathcal{C}$.

3.2. The Taylor tower. Let \mathbf{J} be a small subcategory of \mathbf{Cat} which does not contain the empty category. Let \mathcal{C} and \mathcal{D} be two simplicial model categories. For each $n \geq 0$ we define the endofunctor

$$T_{n,\mathbf{J}} : \operatorname{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \operatorname{Fun}(\mathcal{C}, \mathcal{D})$$

as

$$(T_{n,\mathbf{J}} F)(X) = \operatorname{cholim}_{\mathbf{J}(n+1)} F((-) \star X)$$

We obtain a natural transformation $t_n : Id \Rightarrow T_{n,\mathbf{J}}$. Using the map $\tau_0^{n+1} : \mathbf{J}(n+1) \rightarrow \mathbf{J}(n+2)$ (see below Construction 1.3) we obtain a natural transformation $q_{n+1,1} : T_{n+1,\mathbf{J}} \Rightarrow T_{n,\mathbf{J}}$ such that $q_{n+1,1} t_{n+1,\mathbf{J}} = t_{n,\mathbf{J}}$ for each $n \geq 0$.

Observation 3.1. Let $(\mathcal{V}, \otimes, e)$ be a monoidal category. Let T be an object of \mathcal{V} and $t : e \rightarrow T$ a map. One can construct the functor $\omega \rightarrow \mathcal{V}$

$$e \rightarrow T \cong e \otimes T \xrightarrow{t \otimes T} T \otimes T \cong e \otimes T \otimes T \xrightarrow{t \otimes T \otimes T} T \otimes T \otimes T \cong e \otimes T \otimes T \otimes T \xrightarrow{t \otimes T \otimes T \otimes T} T \otimes T \otimes T \otimes T \cong \dots$$

In particular, let $\operatorname{cst}(e)$ denote the unit object of $\mathcal{V}^{\omega^{op}}$ and let T be an object of $\mathcal{V}^{\omega^{op}}$, with structure maps $q_{n+1,1} : T_{n+1} \rightarrow T_n$. The observation we want to make is that a map $t : \operatorname{cst}(e) \rightarrow T$ gives rise to a functor $\omega \times \omega^{op} \rightarrow \mathcal{V}$, $(i, n) \mapsto T_n^{\otimes i}$, with the convention that for every $X \in \mathcal{V}$, $X^{\otimes 0} = e$.

We shall apply the observation to the situation $\mathcal{V} = \operatorname{END}(\operatorname{Fun}(\mathcal{C}, \mathcal{D}))$, $T_{-,\mathbf{J}}$ defined above (an object of $\mathcal{V}^{\omega^{op}}$) and $t : \operatorname{cst}(Id) \rightarrow T_{-,\mathbf{J}}$ defined above (a map of $\mathcal{V}^{\omega^{op}}$). Then, setting

$$P_{n,\mathbf{J}} = \operatorname{chocolim}_{\omega}(Id \rightarrow T_{n,\mathbf{J}} \rightarrow T_{n,\mathbf{J}}^2 \rightarrow T_{n,\mathbf{J}}^3 \rightarrow \dots)$$

defines an object of \mathcal{V} and there are natural maps $q_{n+1} : P_{n+1,\mathbf{J}} \Rightarrow P_{n,\mathbf{J}}$, so that we obtain an object $P_{-,\mathbf{J}}$ of $\mathcal{V}^{\omega^{op}}$. If $p_n : Id \Rightarrow P_{n,\mathbf{J}}$ is the natural map, then clearly $q_{n+1} p_{n+1} = p_n$, that is, $p : \operatorname{cst}(Id) \Rightarrow P_{-,\mathbf{J}}$.

Definition 3.2. Let \mathcal{C} and \mathcal{D} be two simplicial model categories and $F \in \operatorname{Fun}(\mathcal{C}, \mathcal{D})$. Let \mathbf{J} be a small subcategory of \mathbf{Cat} which does not contain the empty category. $P_{n,\mathbf{J}} F$ is referred to as the n -th **Taylor polynomial** of F with respect to \mathbf{J} , and

$$\dots \rightarrow P_{n,\mathbf{J}} F \xrightarrow{q_n^F} P_{n-1,\mathbf{J}} F \xrightarrow{q_{n-1}^F} \dots \rightarrow P_{1,\mathbf{J}} F \xrightarrow{q_1^F} P_{0,\mathbf{J}} F$$

as the **Taylor tower** of F with respect to \mathbf{J} . We write it as $\{P_{n,\mathbf{J}} F\}$. When \mathbf{J} consists of the terminal category only, $P_{n,*} F$ is referred to as the n -th **Taylor polynomial** of F and written $P_n F$. The Taylor tower of F with respect to $*$ is written $\{P_n F\}$.

Remark 3.3. To construct the Taylor tower of a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ one does not need the full strength of the fact that \mathcal{C} is a simplicial model category. One only needs a category \mathcal{C} with terminal object such that for every finite set S and every $X \in \mathcal{C}$, $S \star X \in \mathcal{C}$. In light of the considerations from 1.2, some full subcategories of a locally presentable pointed simplicial model category consisting of cofibrant objects are natural candidates for such a \mathcal{C} . For example, let \mathcal{C} be a locally λ -presentable pointed simplicial model category such that for every finite set S , tensoring with $N(S_+)^{op}$ preserves λ -presentable objects. We denote by $\mathcal{C}_{\lambda,c}$ the full subcategory of \mathcal{C} consisting of the λ -presentable objects which are cofibrant. Given a simplicial model category \mathcal{D} and a functor $F : \mathcal{C}_{\lambda,c} \rightarrow \mathcal{D}$, one can construct $P_n F : \mathcal{C}_{\lambda,c} \rightarrow \mathcal{D}$.

3.3. Properties of the \star operation. (1) For each $J \in \mathbf{Cat}$, $J \star -$ is a simplicial functor. $J \star -$ preserves the simplicial action provided that \mathcal{C} is pointed.

(2) Let I and J be small categories and $\mathcal{X} : I \rightarrow \mathcal{C}$ a functor. The natural map

$$\mathrm{hocolim}_I(J \star \mathcal{X}(-)) \rightarrow J \star (\mathrm{hocolim}_I \mathcal{X})$$

is an isomorphism provided that \mathcal{C} is pointed. Consequently, in this case one has $I \star (J \star X) \cong J \star (I \star X)$, so that $(T_n F)(J \star (-)) \cong T_n(F(J \star (-)))$ and therefore $(P_n F)(J \star (-)) \cong P_n(F(J \star (-)))$ for every $F \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

(3) Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$ be functors between simplicial model categories. Then

$$(J \star G(-))F = J \star GF(-) : \mathcal{C} \rightarrow \mathcal{E}$$

(4) Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between simplicial model categories. We have a natural transformation $F^{J+} R_J \Rightarrow R_J F$ which induces a natural transformation

$$\mathrm{hocolim}_{J+} F^{J+} R_J(-) \Rightarrow J \star F(-) : \mathcal{C} \rightarrow \mathcal{D}$$

which is a natural isomorphism if $F(*) \cong *$. If, moreover, F is a simplicial functor, then we have a natural transformation

$$\mathrm{hocolim}_{J+} F^{J+} R_J(-) \Rightarrow F(J \star (-)) : \mathcal{C} \rightarrow \mathcal{D}$$

Summing up, if F is a simplicial functor and $F(*) \cong *$, then we have a natural map

$$J \star F(-) \Rightarrow F(J \star (-))$$

(5) Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$ be simplicial functors between simplicial model categories. We have natural transformations $(GF)^{J+} R_J \Rightarrow G^{J+} R_J F \Rightarrow R_J(GF)$ which, together with (4), induce a diagram of natural transformations

$$\begin{array}{ccccc} & \mathrm{hocolim}_{J+}(GF)^{J+} R_J(-) & & & \\ & \swarrow & & \searrow & \\ & J \star GF(-) & & G(\mathrm{hocolim}_{J+} F^{J+} R_J(-)) & \\ & \swarrow & \searrow & \swarrow & \searrow \\ & G(J \star F(-)) & & GF(J \star (-)) & \end{array}$$

If, moreover, $F(*) \cong *$ and $G(*) \cong *$, then by (4) we have natural maps

$$J \star GF(-) \Rightarrow G(J \star F(-)) \Rightarrow GF(J \star (-))$$

3.4. Elementary properties of the T_n and P_n constructions. Let $F \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

(1) Suppose that \mathcal{D} is endowed with simplicial fibrant and cofibrant approximation functors, and that we agree to construct the *cholim* and *chocolim* functors using these approximations. If F is a simplicial functor, then, using the first part of 3.3(1), $T_n F$ and $P_n F$ are simplicial functors as well.

(2) Suppose that the terminal object of \mathcal{C} is cofibrant. If F preserves weak equivalences between cofibrant objects then so do $T_n F$ and $P_n F$.

(3) Let $G \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and let $\alpha : F \Rightarrow G$ be a map which is objectwise a weak equivalence. Then the induced maps $T_n \alpha : T_n F \Rightarrow T_n G$ and $P_n \alpha : P_n F \Rightarrow P_n G$ are objectwise a weak equivalence. Suppose that the terminal object of \mathcal{C} is cofibrant. If α is objectwise a weak equivalence on cofibrant objects, then so are $T_n \alpha$ and $P_n \alpha$.

(4) Let us denote the tensor in \mathcal{C} by $- \otimes -$ and the cotensor in \mathcal{D} by $(-)^{(-)}$. One has

$$T_n F(*) \cong \mathrm{cholim}_{S \in \mathcal{P}_0(\underline{n+1})} F(N(S_+^{op}) \otimes *)$$

If \mathcal{C} is pointed then $T_n F(0) \cong (\hat{F}F0)^{N(\mathcal{P}_0(\underline{n+1}))}$.

(5) Suppose that F preserves weak equivalences. Then for each $X \in \mathcal{C}$, $P_0 F(X)$ has the homotopy type of $F(*)$. Consequently, the map $p_0 F : F \Rightarrow P_0 F$ is an objectwise weak equivalence if and only if $F(X) \rightarrow F(*)$ is a weak equivalence for each $X \in \mathcal{C}$ (if and only if F sends every map in \mathcal{C} to a weak equivalence in \mathcal{D}).

A less elementary property of the T_n and P_n constructions is given in Corollary 5.6.

4. THE AUXILIARY TOWER

Let \mathcal{C} and \mathcal{D} be two simplicial model categories and $F \in \text{Fun}(\mathcal{C}, \mathcal{D})$. In this section we construct a tower $\{\tilde{P}_n F\}$ and a map of towers $\{\tilde{P}_n F\} \rightarrow \{P_n F\}$. If the terminal object of \mathcal{C} is cofibrant and F preserves weak equivalences between cofibrant objects, this map is shown to be a weak equivalence when evaluated at cofibrant objects.

4.1. The \star^h operation. Let \mathcal{C} be a simplicial model category with simplicial action which we denote by \otimes . Let S be a set, viewed as a discrete category and $X \in \mathcal{C}$. Recall from 3.1 the functor R_S . We define a functor $(S, X) : (S_+)^{op} \times S_+ \rightarrow \mathcal{C}$ as $(S, X)(j, k) = N(j \downarrow S_+)^{op} \otimes R_S X(k)$. From 1.2 and 3.1 we have

$$\text{colim}_{\int_{po} S} (S, X) K \xi_S \xrightarrow{\cong} \int_{j \in S_+} (S, X) \cong S * X$$

For simplicity, we define $\tilde{R}_S : \mathcal{C} \rightarrow \mathcal{C}_{po}^{\int S}$ as $\tilde{R}_S X = (S, X) K \xi_S$. Precisely, $\tilde{R}_S X(0, k) = N(k \downarrow S_+)^{op} \otimes * = *$, $\tilde{R}_S X(1, j) = N(j \downarrow S_+)^{op} \otimes X = X$ and $\tilde{R}_S X(2, *) = N(S_+)^{op} \otimes X$. By definition

$$S \star^h X = \text{hocolim}_{\int_{po} S} \tilde{R}_S X$$

4.2. The auxiliary tower. Let \mathcal{C} and \mathcal{D} be two simplicial model categories. For each $n \geq 0$ we define the endofunctor

$$\tilde{T}_n : \text{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D})$$

as

$$(\tilde{T}_n F)(X) = \text{cholim}_{\mathcal{P}_0(\underline{n+1})} F((-) \star^h X)$$

We obtain a natural transformation $\tilde{t}_n : Id \Rightarrow \tilde{T}_n$. Using the map $\tau_0^{n+1} : \mathcal{P}_0(\underline{n+1}) \rightarrow \mathcal{P}_0(\underline{n+2})$ (Example 1.4) we obtain a natural transformation $\tilde{q}_{n+1,1} : \tilde{T}_{n+1} \Rightarrow \tilde{T}_n$ such that $\tilde{q}_{n+1,1} \tilde{t}_{n+1} = \tilde{t}_n$ for each $n \geq 0$. We apply Observation 3.1 to the situation $\mathcal{V} = \text{END}(\text{Fun}(\mathcal{C}, \mathcal{D}))$, \tilde{T} defined above (an object of $\mathcal{V}^{\omega^{op}}$) and $\tilde{t} : \text{cst}(Id) \rightarrow \tilde{T}$ defined above (a map of $\mathcal{V}^{\omega^{op}}$). Then, setting

$$\tilde{P}_n = \text{chocolim}_{\omega}(Id \rightarrow \tilde{T}_n \rightarrow \tilde{T}_n^2 \rightarrow \tilde{T}_n^3 \rightarrow \dots)$$

defines an object of \mathcal{V} and there are natural maps $\tilde{q}_{n+1} : \tilde{P}_{n+1} \Rightarrow \tilde{P}_n$, so that we obtain an object \tilde{P} of $\mathcal{V}^{\omega^{op}}$. If $\tilde{p}_n : Id \Rightarrow \tilde{P}_n$ is the natural map, then clearly $\tilde{q}_{n+1} \tilde{p}_{n+1} = \tilde{p}_n$, that is, $\tilde{p} : \text{cst}(Id) \Rightarrow \tilde{P}$.

Lemma 4.1. *Suppose that the terminal object of \mathcal{C} is cofibrant and $F \in \text{Fun}(\mathcal{C}, \mathcal{D})$ preserves weak equivalences between cofibrant objects. Then the natural map of towers $\{\tilde{P}_n F\} \rightarrow \{P_n F\}$ is a weak equivalence when evaluated at cofibrant objects.*

Proof. Let $X \in \mathcal{C}$ be cofibrant. By hypothesis, it suffices to show that for every finite set S , the map $S \star^h X \rightarrow S \star X$ is a weak equivalence. For this, it suffices by [6, 19.9.1(1)] to show that $\tilde{R}_S X$ is Reedy cofibrant. $\tilde{R}_S X$ is a spider in \mathcal{C} with $|S|$ legs (Example 1.5) whose body is $N(S_+)^{op} \otimes X$. It is cofibrant since the natural map $\coprod_S X \rightarrow N(S_+)^{op} \otimes X$ is a cofibration. \square

It seems natural to us to relate Rezk's proof [11] of Goodwillie's n -excisive approximation theorem [5, Theorem 1.8] to the tower $\{\tilde{P}_n F\}$. Let $\mathcal{X} : \mathcal{P}(\underline{n+1}) \rightarrow \mathcal{C}$. For $U, T \in \mathcal{P}(\underline{n+1})$ we have

$$U \star^h \mathcal{X}(T) = \text{hocolim}_{\int_{po} U} \tilde{R}_U \mathcal{X}(T)$$

Following Rezk, we define $\mathcal{X}^2 : \mathcal{P}(\underline{n+1}) \times \mathcal{P}(\underline{n+1}) \rightarrow \mathcal{C}$ as

$$\mathcal{X}^2(U, T) = \text{hocolim}_{\int_{po} U} \underline{\mathcal{X}}_T$$

where $\underline{\mathcal{X}}_T(0, s) = \mathcal{X}(T \cup \{s\})$, $\underline{\mathcal{X}}_T(1, s) = \mathcal{X}(T)$ and $\underline{\mathcal{X}}_T(2, *) = \mathcal{X}(T)$. We have natural maps

$$\mathcal{X}^2(U, T) \longrightarrow U \star^h \mathcal{X}(T)$$

and

$$\mathcal{X}_U(T) \stackrel{\text{def}}{=} \text{hocolim}(\mathcal{X}(T) \leftarrow \coprod_U \mathcal{X}(T) \rightarrow \coprod_U \mathcal{X}(T \cup \{s\})) \longrightarrow \mathcal{X}^2(U, T)$$

5. HOMOTOPY LIMITS AND COLIMITS II

This section is a continuation of section 2. We review here more elaborated results on (corrected) homotopy limits and colimits in simplicial model categories.

5.1. Homotopy limits. Recall that, in a simplicial model category, a diagram

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

with X, Y and Z fibrant objects is homotopy Cartesian in the sense of [4, Definition 1.3] if and only if it is a homotopy pullback in the sense of [7, Chapter 7].

Let \mathcal{C} be a pointed simplicial model category. If $g : X \rightarrow Y$ is a map in \mathcal{C} , we denote by $\text{chf}(g)$ the corrected homotopy limit of the diagram $X \xrightarrow{g} Y \leftarrow 0$. This defines a functor $\text{chf} : \mathcal{C}^{[1]} \rightarrow \mathcal{C}$. For each small category J , there is then a functor $\text{chf} : (\mathcal{C}^J)^{[1]} \rightarrow \mathcal{C}^J$ given as $\text{hf}(g)_j = \text{chf}(g_j)$. We denote by $\text{hf}(g)$ the homotopy limit of the diagram $X \xrightarrow{g} Y \leftarrow 0$. This defines a functor $\text{hf} : \mathcal{C}^{[1]} \rightarrow \mathcal{C}$.

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between pointed simplicial model categories. We have maps

$$F\text{chf}(g) \rightarrow FX \times_{FY}^{ch} F0 \leftarrow \text{chf}(Fg)$$

where $FX \times_{FY}^{ch} F0$ is the corrected homotopy limit of the diagram $FX \xrightarrow{Fg} FY \leftarrow F0$. The two maps displayed above are weak equivalences if, for example, $F0 \rightarrow 0$ is a weak equivalence and F sends homotopy pullback diagrams of the form

$$\begin{array}{ccc} A & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ B & \longrightarrow & C \end{array}$$

to homotopy pullbacks.

The next result is closely related to [12, Lemma 1.3.2(c)].

Lemma 5.1. *Let I be a small filtered category and \mathcal{C} a locally finitely presentable simplicial model category whose tensor, viewed as a functor $\mathbf{S} \times \mathcal{C} \rightarrow \mathcal{C}$, preserves finitely presentable objects, and such that an I -indexed colimit of weak equivalences of \mathcal{C} is a weak equivalence and an I -indexed colimit of fibrant objects of \mathcal{C} is fibrant. Let J be a finite category such that for each object j of J , the nerve of $(J \downarrow j)$ is finitely presentable. Then for every $\mathcal{X} : I \rightarrow \mathcal{C}^J$, the natural map*

$$\text{colim}_I \text{cholim}_J \mathcal{X} \rightarrow \text{cholim}_J(\text{colim}_I \mathcal{X})$$

is a weak equivalence.

Proof. The natural map is the composite

$$\text{colim}_I \text{cholim}_J \mathcal{X} = \text{colim}_I \text{holim}_J \hat{F}\mathcal{X} \xrightarrow{\cong} \text{holim}_J(\text{colim}_I \hat{F}\mathcal{X}) \rightarrow \text{holim}_J \hat{F}(\text{colim}_I \mathcal{X}) = \text{cholim}_J(\text{colim}_I \mathcal{X})$$

The first map is an isomorphism by Lemma 5.2. The second map can be seen to be a weak equivalence using the commutative diagram

$$\begin{array}{ccc} \text{colim}_I \hat{F}\mathcal{X} & \xrightarrow{\quad} & \hat{F}(\text{colim}_I \mathcal{X}) \\ & \nwarrow \quad \nearrow & \\ & \text{colim}_I \mathcal{X} & \end{array}$$

and the other assumptions on \mathcal{C} . □

Lemma 5.2. *Let \mathcal{C} be a locally finitely presentable simplicial model category whose tensor, viewed as a functor $\mathbf{S} \times \mathcal{C} \rightarrow \mathcal{C}$, preserves finitely presentable objects. Let J be a finite category such that for each object j of J , the nerve of $(J \downarrow j)$ is finitely presentable. Then holim_J preserves filtered colimits.*

Proof. The fact that \mathcal{C} is a model category is not relevant. Perhaps the simplest proof is to write holim_J as a limit indexed over the (upper) twisted arrow category of J and to use adjunctions and standard properties of locally presentable categories. □

We recall [8, Lemma 4.3] that, in an almost finitely generated model category, an ω -indexed colimit of weak equivalences is a weak equivalence and an ω -indexed colimit of fibrant objects is fibrant.

Let now $p : \mathbb{E} \rightarrow \mathbb{B}$ a split fibration between small categories. We denote by \mathbb{E}_b the fiber category over $b \in \mathbb{B}$ and by $\iota_b : \mathbb{E}_b \rightarrow \mathbb{E}$ the natural functor. Let \mathcal{C} be a simplicial model category and $\mathcal{X} : \mathbb{E} \rightarrow \mathcal{C}$ a functor. We obtain a functor

$$\mathbb{B} \rightarrow (\mathbf{Cat} // \mathcal{C}), \quad b \mapsto \mathcal{X}_b := \mathcal{X} \iota_b$$

Therefore there is a natural map

$$\text{cholim}_{\mathbb{E}} \mathcal{X} \rightarrow \text{holim}_{\mathbb{B}} \text{cholim}_{\mathbb{E}_b} \mathcal{X}_b$$

For $b \in \mathbb{B}$ we denote by q the natural functor $(b \downarrow p) \rightarrow \mathbb{E}$.

Theorem 5.3. [2, A dual of theorem 26.8] *Let $p : \mathbb{E} \rightarrow \mathbb{B}$ a split fibration between small Reedy categories with cofibrant constants such that (i) p is a morphism of Reedy categories, (ii) (p^*, p_*) is a Quillen pair and (iii) for each $b \in \mathbb{B}$, $(q_!, q^*)$ is a Quillen pair. Let \mathcal{C} be a simplicial model category and $\mathcal{X} : \mathbb{E} \rightarrow \mathcal{C}$ a functor. Then the natural map*

$$\text{cholim}_{\mathbb{E}} \mathcal{X} \rightarrow \text{holim}_{\mathbb{B}} \text{cholim}_{\mathbb{E}_b} \mathcal{X}_b$$

is a weak equivalence.

Theorem 5.3 will be applied to split fibrations with base pb .

5.2. Homotopy colimits. Let I be a small filtered category and $\phi : I \rightarrow \mathbf{Cat}$ a functor. Let $J = \text{colim}_I \phi$ and let $u_i : \phi_i \rightarrow J$ be the canonical map ($i \in I$). If $\mathcal{X} : J \rightarrow \mathcal{C}$ is a functor, then [1, XII 3.5]

$$\text{hocolim}_J \mathcal{X} \cong \text{colim}_I (\text{hocolim}_{\phi_i} \mathcal{X}/i)$$

where \mathcal{X}/i is the composite $\phi_i \xrightarrow{u_i} J \xrightarrow{\mathcal{X}} \mathcal{C}$. In particular, let J be a small Reedy category and $F^i J$ the i -filtration of J [6, 15.1.22]. Consider $\phi : \omega \rightarrow \mathbf{Cat}$, $\phi_i = F^i J$. Then $J = \text{colim}_{\omega} \phi$ [6, 15.1.25], hence $\text{hocolim}_J \mathcal{X} \cong \text{colim}_{\omega} (\text{hocolim}_{F^i J} \mathcal{X}/i)$.

Lemma 5.4. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between simplicial model categories which preserves weak equivalences. Let I be a small filtered category such that an I -indexed colimit of weak equivalences of \mathcal{D} is a weak equivalence. Suppose that F preserves I -indexed colimits. Then for every $\mathcal{X} : I \rightarrow \mathcal{C}$, $\text{chocolim}_I F\mathcal{X}$ and $F(\text{chocolim}_I \mathcal{X})$ are weakly equivalent. If $\mathcal{C} = \mathcal{D}$ and there is a natural transformation $\text{Id} \Rightarrow F$ which is objectwise a weak equivalence, the requirement that F preserves I -indexed colimits can be dropped.*

Proof. For the first part we have chains of arrows

$$\text{hocolim}_I \tilde{C}F\mathcal{X} \cong \text{colim}_I (\text{hocolim}_{(I \downarrow i)} \tilde{C}F\mathcal{X}/i) \rightarrow \text{colim}_I \tilde{C}F\mathcal{X}(i) \rightarrow \text{colim}_I F\mathcal{X}(i) \leftarrow \text{colim}_I F\tilde{C}\mathcal{X}(i)$$

and

$$F(\text{hocolim}_I \tilde{C}\mathcal{X}) \cong F(\text{colim}_I (\text{hocolim}_{(I \downarrow i)} \tilde{C}\mathcal{X}/i)) \xrightarrow{\cong} \text{colim}_I F(\text{hocolim}_{(I \downarrow i)} \tilde{C}\mathcal{X}/i) \rightarrow \text{colim}_I F\tilde{C}\mathcal{X}(i)$$

Each arrow in the above chains is a weak equivalence by [6, 19.6.8(1)] and hypothesis. For the second part, the isomorphism in the second chain of arrows is a weak equivalence. \square

Lemma 5.4 gives sufficient conditions for a functor to be ‘ I -finitary’ [5, Definition 5.10]. The next result, which in common parlance says that certain homotopy limits ‘commute’ with filtered homotopy colimits, is a consequence of Lemmas 5.4 and 5.1.

Corollary 5.5. *Let I be a small filtered category and \mathcal{C} a locally finitely presentable simplicial model category whose tensor, viewed as a functor $\mathbf{S} \times \mathcal{C} \rightarrow \mathcal{C}$, preserves finitely presentable objects, and that an I -indexed colimit of weak equivalences of \mathcal{C} is a weak equivalence and an I -indexed colimit of fibrant objects of \mathcal{C} is fibrant. Let J be a finite category such that for each object j of J , the nerve of $(J \downarrow j)$ is finitely presentable. Then for every $\mathcal{X} : I \rightarrow \mathcal{C}^J$, $\text{chocolim}_J \text{cholim}_I \mathcal{X}$ and $\text{cholim}_J \text{chocolim}_I \mathcal{X}$ are weakly equivalent.*

Let \mathcal{C} and \mathcal{D} be two simplicial model categories, I a small category and $F : I \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D})$. We define $\text{chocolim}_I F : \mathcal{C} \rightarrow \mathcal{D}$ as $(\text{chocolim}_I F)(X) = \text{chocolim}_I F(X)$.

Corollary 5.6. *Let \mathcal{C} and \mathcal{D} be two simplicial model categories and I a small filtered category. Suppose that \mathcal{D} is locally presentable, that its tensor, viewed as a functor $\mathbf{S} \times \mathcal{D} \rightarrow \mathcal{D}$, preserves finitely presentable objects, and that an I -indexed colimit of weak equivalences of \mathcal{D} is a weak equivalence and an I -indexed colimit of fibrant objects of \mathcal{D} is fibrant. Then, for every $F : I \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D})$, $T_n(\text{chocolim}_I F)$ and $\text{chocolim}_I T_n F$ are objectwise weakly equivalent. Consequently, $P_n(\text{chocolim}_I F)$ and $\text{chocolim}_I P_n F$ are objectwise weakly equivalent.*

6. GENERALIZED HOMOTOPY CARTESIAN CUBES

In this section we give an analogue, in our context, of some useful results from [4],[9], [3] and [5] which revolve around the notion of homotopy Cartesian cube [4, Definition 1.3].

Definition 6.1. *Let \mathcal{C} be a simplicial model category and J a small category. An object \mathcal{X} of \mathcal{C}^{J+} is **homotopy Cartesian** if the natural map*

$$\mathcal{X}(\emptyset) \rightarrow \text{cholim}_J \mathcal{X}$$

is a weak equivalence.

Elementary facts about homotopy Cartesian objects 6.2. (1) *Let $\mathcal{X} : J \rightarrow \mathcal{C}$. Define $R\mathcal{X} : J_+ \rightarrow \mathcal{C}$ as*

$$R\mathcal{X}(j) = \begin{cases} \text{cholim}_J \mathcal{X}, & \text{if } j = \emptyset \\ \hat{F}\mathcal{X}(j), & \text{otherwise} \end{cases}$$

Then $R\mathcal{X}$ is homotopy Cartesian.

(2) *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between simplicial model categories and J a small category. Suppose that F preserves weak equivalences and homotopy Cartesian objects $J_+ \rightarrow \mathcal{C}$. Then for any $\mathcal{X} : J \rightarrow \mathcal{C}$, $F(\text{cholim}_J \mathcal{X})$ and $\text{cholim}_J F\mathcal{X}$ are weakly equivalent.*

(3) *Let $\mathcal{X}, \mathcal{Y} : J_+ \rightarrow \mathcal{C}$ be two functors. If $\mathcal{X} \rightarrow \mathcal{Y}$ is an objectwise weak equivalence, then \mathcal{X} is homotopy Cartesian if and only if \mathcal{Y} is.*

(4) *Suppose that J a small Reedy category with cofibrant constants. Let $\mathcal{X} \in \mathcal{C}^{J+}$ be a fibrant object. Then [6, 19.9.1(2)] the natural map $\lim_J \mathcal{X} \rightarrow \text{holim}_J \mathcal{X}$ is a weak equivalence, so in this case \mathcal{X} is homotopy Cartesian if and only if $\mathcal{X}(\emptyset) \rightarrow \lim_J \mathcal{X}$ is a weak equivalence.*

Replacement 6.3. [4, Remark 1.11] *Let J be a small Reedy category with cofibrant constants and such that every map in the direct subcategory \vec{J} is a monomorphism. The natural functor $J \rightarrow J_+$ is an inclusion, let's call it u . The functor $u^* : \mathcal{C}^{J+} \rightarrow \mathcal{C}^J$ has a (full and faithful) right adjoint u_* calculated as*

$$u_* \mathcal{X}(j) = \begin{cases} \lim_J \mathcal{X}, & \text{if } j = \emptyset \\ \mathcal{X}(j), & \text{otherwise} \end{cases}$$

Let $\mathcal{X} : J \rightarrow \mathcal{C}$ be a fibrant object. For each $j \in J$ we denote by Q_j the composite $(j \downarrow J) \rightarrow J \xrightarrow{\mathcal{X}} \mathcal{C}$. Define $\mathcal{Z} : J_+ \rightarrow \mathcal{C}$ as

$$\mathcal{Z}(j) = \begin{cases} \text{holim}_J \mathcal{X}, & \text{if } j = \emptyset \\ \text{holim}_{(j \downarrow J)} Q_j, & \text{otherwise} \end{cases}$$

Then \mathcal{Z} is homotopy Cartesian and the natural map $\mathcal{X} \rightarrow u^ \mathcal{Z}$ is an objectwise weak equivalence.*

Let \mathcal{C} be a simplicial model category, J a small category and $\mathcal{X} : (\int J)_+ \rightarrow \mathcal{C}$. Using the isomorphism (3) from Example 1.2 we can identify \mathcal{X} with an arrow $\mathcal{X}_{\text{left}} \rightarrow \mathcal{X}_{\text{right}}$ of \mathcal{C}^{J+} . If J a Reedy category with cofibrant constants we have a homotopy pullback diagram

$$\begin{array}{ccc} \text{cholim}_{\int J} \mathcal{X} & \xrightarrow{\quad} & \hat{F}\mathcal{X}_{\text{right}}(\emptyset) \\ \downarrow \text{pb} & & \downarrow \\ \text{cholim}_J \mathcal{X}_{\text{left}} & \xrightarrow{\quad} & \text{cholim}_J \mathcal{X}_{\text{right}} \end{array}$$

Lemma 6.4. [4, Proposition 1.6] *Let \mathcal{C} be a simplicial model category and J a small Reedy category with cofibrant constants. Let $\mathcal{X} : (\int J)_+ \rightarrow \mathcal{C}$ be a functor.*

- (i) *If $\mathcal{X}_{\text{left}}$ and $\mathcal{X}_{\text{right}}$ are homotopy Cartesian then so is \mathcal{X} .*
- (ii) *If \mathcal{X} and $\mathcal{X}_{\text{right}}$ are homotopy Cartesian then so is $\mathcal{X}_{\text{left}}$.*

Let \mathcal{C} be a pointed simplicial model category and J a small category. We denote by ∂ the composite functor

$$\mathcal{C}^{(\int J)_+}_{pb} \cong (\mathcal{C}^{J_+})^{[1]} \xrightarrow{\text{chf}} \mathcal{C}^{J_+}$$

so that $\partial\mathcal{X}(\emptyset) = \text{chf}(\mathcal{X}_{\text{left}}(\emptyset) \rightarrow \mathcal{X}_{\text{right}}(\emptyset))$ and $\partial\mathcal{X}(j) = \text{chf}(\mathcal{X}_{\text{left}}(j) \rightarrow \mathcal{X}_{\text{right}}(j))$. It follows that

$$\text{holim}_J \partial\mathcal{X} \cong \text{hf}(\text{cholim}_J \mathcal{X}_{\text{left}} \rightarrow \text{cholim}_J \mathcal{X}_{\text{right}})$$

If $\mathcal{X}, \mathcal{Y} : (\int J)_+ \rightarrow \mathcal{C}$ are two functors and $\mathcal{X} \rightarrow \mathcal{Y}$ is an objectwise weak equivalence, then $\partial\mathcal{X} \rightarrow \partial\mathcal{Y}$ is an objectwise weak equivalence.

Lemma 6.5. [9, Lemma 4.7] *Let \mathcal{C} be a pointed simplicial model category and J a small Reedy category with cofibrant constants. Let $\mathcal{X} : (\int J)_+ \rightarrow \mathcal{C}$ be a functor. If \mathcal{X} is homotopy Cartesian then so is $\partial\mathcal{X}$. If \mathcal{C} is moreover a stable model category, then the converse holds.*

Lemma 6.6. *Let J be a small category and $F : \mathcal{C} \rightarrow \mathcal{D}$ a functor between pointed simplicial model categories such that*

- (i) *$F0 \rightarrow 0$ is a weak equivalence, and*
- (ii) *F sends homotopy pullback diagrams of the form*

$$\begin{array}{ccc} A & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ B & \longrightarrow & C \end{array}$$

to homotopy pullbacks.

Let $\mathcal{X} : (\int J)_+ \rightarrow \mathcal{C}$ be a functor. Then there are a functor $H : J_+ \rightarrow \mathcal{D}$ and a diagram of functors $F\partial\mathcal{X} \rightarrow H \leftarrow \partial F\mathcal{X}$ in which every map is an objectwise weak equivalence.

Proof. This follows from the considerations in 2.1. □

Corollary 6.7. [3, Proof of Lemma 1.19] *Let J be a small Reedy category with cofibrant constants and \mathcal{C} and \mathcal{D} pointed simplicial model categories with \mathcal{D} stable. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor such that*

- (i) *$F0 \rightarrow 0$ is a weak equivalence, and*
- (ii) *F sends homotopy pullback diagrams of the form*

$$\begin{array}{ccc} A & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ B & \longrightarrow & C \end{array}$$

to homotopy pullbacks.

If F preserves homotopy Cartesian objects $J_+ \rightarrow \mathcal{C}$, then F preserves homotopy Cartesian objects $(\int J)_+ \rightarrow \mathcal{C}$.

Proof. One uses Lemmas 6.5 and 6.6, and 6.2(3). □

Lemma 6.8. *Let \mathcal{C} be a locally finitely presentable simplicial model category whose tensor, viewed as a functor $\mathbf{S} \times \mathcal{C} \rightarrow \mathcal{C}$, preserves finitely presentable objects, and such that an ω -indexed colimit of weak equivalences of \mathcal{C} is a weak equivalence and an ω -indexed colimit of fibrant objects of \mathcal{C} is fibrant. Let J be a finite category such that for each object j of J , the nerve of $(J \downarrow j)$ is finitely presentable. Then a sequential corrected homotopy colimit of homotopy Cartesian objects of \mathcal{C}^{J_+} is homotopy Cartesian.*

Proof. This is a consequence of Corollary 5.5. □

7. ON (STRONGLY) HOMOTOPY CO-CARTESIAN CUBES

In this section we give a model theoretic interpretation of the notions of homotopy co-Cartesian and strongly homotopy co-Cartesian cube from [4, Definitions 1.4 and 2.1].

Let \mathcal{C} be a fixed model category. Let J be a small Reedy category. Following [6, Chapter 15], we denote by $F^n J$ the n -filtration of J and by $I^n : F^{n-1} J \rightarrow F^n J$ the inclusion functor. An object of \mathcal{C}^J is cofibrant if and only if for every $n \geq 0$, its restriction to $F^n J$ is cofibrant. For each object α of J of degree n , $(I^n \downarrow \alpha)$ is a Reedy category. The restriction functor $I^{n*} : \mathcal{C}^{F^n J} \rightarrow \mathcal{C}^{F^{n-1} J}$ has a left adjoint $I_!^n$ and a right adjoint I_*^n .

Lemma 7.1. *Suppose that J has fibrant constants. Then both $F^n J$ and $(I^n \downarrow \alpha)$ have fibrant constants, where α is an object of J of degree n , and $(I_!^n, I_*^n)$ is a Quillen pair.*

The functor I^{n*} is a cloven Grothendieck bifibration. The fiber category of I^{n*} over $\mathcal{X} \in \mathcal{C}^{F^{n-1} J}$ is denoted by $(\mathcal{C}^{F^n J})_{\mathcal{X}}$. A cartesian lift of $u : \mathcal{Y} \rightarrow I^{n*} \mathcal{Z}$ is given by the pullback diagram

$$\begin{array}{ccc} u^* \mathcal{Z} & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ I_*^n \mathcal{Y} & \longrightarrow & I_*^n I^{n*} \mathcal{Z} \end{array}$$

A cocartesian lift of $u : I^{n*} \mathcal{Z} \rightarrow \mathcal{Y}$ is given by the pushout diagram

$$\begin{array}{ccc} I_!^n I^{n*} \mathcal{Z} & \longrightarrow & I_!^n \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{Z} & \longrightarrow & u_! \mathcal{Z} \end{array}$$

Every map $f : \mathcal{X} \rightarrow \mathcal{Y}$ of $\mathcal{C}^{F^n J}$ can be decomposed as $\mathcal{X} \xrightarrow{f^u} u^*(\mathcal{Y}) \xrightarrow{\text{cart}} \mathcal{Y}$, where $u = p(f)$ and $\text{cart} : u^*(\mathcal{Y}) \rightarrow \mathcal{Y}$ is cartesian over u , and as $\mathcal{X} \xrightarrow{\text{cocart}} u_!(\mathcal{X}) \xrightarrow{f_u} \mathcal{Y}$, where $\text{cocart} : \mathcal{X} \rightarrow u_! \mathcal{X}$ is cocartesian over u .

Lemma 7.2. *For every object \mathcal{X} of $\mathcal{C}^{F^{n-1} J}$, $(\mathcal{C}^{F^n J})_{\mathcal{X}}$ has a model structure in which a map f is a weak equivalence, cofibration or fibration if f is so in $\mathcal{C}^{F^n J}$.*

Proof. The factorization axiom of a model category is proved inductively on the degree of the objects of \mathcal{C} , exactly as in [6, 15.3.16]. The difference with *loc. cit.* is in degrees $\leq n-1$, when we choose the factorization to be given by identity maps. \square

Theorem 7.3. *Let J be a small Reedy category with fibrant constants. For each $n \geq 1$ the category $\mathcal{C}^{F^n J}$ admits a model structure in which a map $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a*

- *weak equivalence if $I^{n*}(f)$ is a weak equivalence in $\mathcal{C}^{F^{n-1} J}$;*
- *cofibration if f_u is a trivial cofibration in $(\mathcal{C}^{F^n J})_{I^{n*} \mathcal{Y}}$ and $I^{n*}(f)$ is a cofibration in $\mathcal{C}^{F^{n-1} J}$;*
- *fibration if f^u is a fibration in $(\mathcal{C}^{F^n J})_{I^{n*} \mathcal{X}}$ and $I^{n*}(f)$ is a fibration in $\mathcal{C}^{F^{n-1} J}$.*

We denote this model structure by $L_r \mathcal{C}^{F^n J}$. $L_r \mathcal{C}^{F^n J}$ is a right Bousfield localization of $\mathcal{C}^{F^n J}$. The adjoint pair

$$I_!^n : \mathcal{C}^{F^{n-1} J} \rightleftarrows L_r \mathcal{C}^{F^n J} : I^{n*}$$

is a Quillen equivalence.

Proof. This is an application of Theorem 7.4, using Lemmas 7.1 and 7.2. \square

The next result is a consequence of [13, 2.2].

Theorem 7.4. *Let $p : \mathbb{E} \rightarrow \mathbb{B}$ be a cloven bifibration. Suppose that*

- (i) *the base category \mathbb{B} has a model structure $(\mathcal{Cof}, \mathcal{W}, \mathcal{Fib})$;*
- (ii) *for each object I of \mathbb{B} , the fibre category \mathbb{E}_I has a model structure $(\mathcal{Cof}_I, \mathcal{W}_I, \mathcal{Fib}_I)$;*
- (iii) *for every morphism $u : I \rightarrow J$ of \mathbb{B} , we have $u^*(\mathcal{Fib}_J) \subseteq \mathcal{Fib}_I$ and $u^*(\mathcal{Fib}_J \cap \mathcal{W}_J) \subseteq \mathcal{Fib}_I \cap \mathcal{W}_I$.*

Then \mathbb{E} has a model structure in which a map f is a

- *weak equivalence if $p(f)$ is a weak equivalence in \mathbb{B} ;*
- *cofibration if $f_u \in \mathcal{Cof}_{p(\text{cod}(f))} \cap \mathcal{W}_{p(\text{cod}(f))}$ and $p(f) \in \mathcal{Cof}$;*
- *fibration if $f^u \in \mathcal{Fib}_{p(\text{dom}(f))}$ and $p(f) \in \mathcal{Fib}$.*

We spell out that an object \mathcal{X} of $\mathcal{C}^{F^n J}$ is cofibrant in $L_r \mathcal{C}^{F^n J}$ if and only if $I^{n*} \mathcal{X}$ is cofibrant in $\mathcal{C}^{F^{n-1} J}$ and $I^n I^{n*} \mathcal{X} \rightarrow \mathcal{X}$ is a trivial cofibration in $\mathcal{C}^{F^n J}$ if and only if $I^{n*} \mathcal{X}$ is cofibrant in $\mathcal{C}^{F^{n-1} J}$ and for each object α of $F^n J$ of degree n , the canonical map $\text{colim}_{(I^n \downarrow \alpha)} I^{n*} \mathcal{X} \rightarrow \mathcal{X}(\alpha)$ is a trivial cofibration. By [6, 15.2.9] this is equivalent to saying that \mathcal{X} is cofibrant in $\mathcal{C}^{F^n J}$ and for each object α of $F^n J$ of degree n , the latching map $L_\alpha \mathcal{X} \rightarrow \mathcal{X}(\alpha)$ of \mathcal{X} at α is a weak equivalence.

Suppose now that \mathcal{C} is a simplicial model category. Let $\mathcal{X} : J \rightarrow \mathcal{C}$. For each object α of J of degree n we have a commutative diagram

$$\begin{array}{ccc} (c)\text{hocolim}_{\partial(\vec{J} \downarrow \alpha)} \mathcal{X} & \longrightarrow & \text{colim}_{\partial(\vec{J} \downarrow \alpha)} \mathcal{X} = L_\alpha \mathcal{X} \longrightarrow \mathcal{X}(\alpha) \\ \downarrow & & \downarrow \cong \\ (c)\text{hocolim}_{(I^n \downarrow \alpha)} \mathcal{X} & \longrightarrow & \text{colim}_{(I^n \downarrow \alpha)} \mathcal{X} \end{array}$$

Example 7.5. Let $n \geq 2$. The category $\mathcal{P}(\underline{n})$ (Example 1.4) becomes a Reedy category with fibrant constants if we let the direct subcategory $\overrightarrow{\mathcal{P}}(\underline{n})$ be $\mathcal{P}(\underline{n})$, the inverse subcategory $\overleftarrow{\mathcal{P}}(\underline{n})$ be the discrete category on the set of objects of $\mathcal{P}(\underline{n})$ and the degree of $S \in \mathcal{P}(\underline{n})$ be $|S|$. One has $F^0 \mathcal{P}(\underline{n}) = *$, $F^1 \mathcal{P}(\underline{n}) = \underline{n}_+$, $F^{n-1} \mathcal{P}(\underline{n}) = \mathcal{P}_1(\underline{n})$ and $F^n \mathcal{P}(\underline{n}) = \mathcal{P}(\underline{n})$. If $S \subset \underline{n}$ has cardinality k , $\partial(\overrightarrow{\mathcal{P}}(\underline{n}) \downarrow S) = (I^k \downarrow S) \cong \mathcal{P}_1(S)$. The cofibrant objects of $\mathcal{C}^{\mathcal{P}(\underline{n})}$ are the cofibration cubes of [4, Definition 1.13]. The cofibrant objects of $L_r \mathcal{C}^{\mathcal{P}(\underline{n})}$ are the cofibration cubes \mathcal{X} for which $\text{colim}_{\mathcal{P}_1(\underline{n})} \mathcal{X} \rightarrow \mathcal{X}(\underline{n})$ is a weak equivalence. In general, for $k \geq 2$, the cofibrant objects of $L_r \mathcal{C}^{F^k \mathcal{P}(\underline{n})}$ are the cofibrant objects \mathcal{X} for which $\text{colim}_{\mathcal{P}_1(S)} \mathcal{X} \rightarrow \mathcal{X}(S)$ is a weak equivalence for each subset S of \underline{n} with $|S| = k$. If \mathcal{C} is moreover a simplicial model category, by [6, 19.9.1(1)] the cofibrant objects of $L_r \mathcal{C}^{\mathcal{P}(\underline{n})}$ are the cofibration cubes \mathcal{X} for which $\text{hocolim}_{\mathcal{P}_1(\underline{n})} \mathcal{X} \rightarrow \mathcal{X}(\underline{n})$ is a weak equivalence, and for $k \geq 2$, the cofibrant objects of $L_r \mathcal{C}^{F^k \mathcal{P}(\underline{n})}$ are the cofibrant objects \mathcal{X} for which $\text{hocolim}_{\mathcal{P}_1(S)} \mathcal{X} \rightarrow \mathcal{X}(S)$ is a weak equivalence for each subset S of \underline{n} with $|S| = k$. Using the dual of Lemma 6.4 and properties of homotopy pushouts, it follows that a cofibrant object \mathcal{X} of $\mathcal{C}^{\mathcal{P}(\underline{n})}$ is cofibrant in $L_r \mathcal{C}^{F^k \mathcal{P}(\underline{n})}$ for each $2 \leq k \leq n$ if and only if \mathcal{X} is strongly homotopy co-Cartesian in the sense of [4, Definition 2.1].

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